# Week 7 Inner Product Space

1st Half: Normed Space

Distance $\rightsquigarrow$ Topology

(Not enough to do geometry)

**Motivation:**

Dot Product in $\mathbb{R}^n$ $\rightsquigarrow$ length, angle, perpendicularity, Projection

$\vec{x}\cdot\vec{y} = \sum_{i=1}^{n} x_i y_i$

In $\mathbb{C}$ $\quad z = a+bi \quad \bar{z} = a-bi \quad a,b\in\mathbb{R}$

length of $z = \sqrt{z\bar{z}}$

Generalize these to vector space

**Defn** Let $X$ be a vector space $/\mathbb{F}$ ($\mathbb{R}$ or $\mathbb{C}$)

An inner product is a function

$$X \times X \longrightarrow \mathbb{F}$$

$$(x,y) \longmapsto \langle x,y\rangle$$

$\leftarrow$ real or complex number

Such that $\forall x,y,z \in X$ and $\alpha \in \mathbb{F}$

IP1 : $\langle x+y, z\rangle = \langle x,z\rangle + \langle y,z\rangle$ } linear in 1st component

IP2 : $\langle \alpha x, y\rangle = \alpha\langle x,y\rangle$

IP3 : $\langle x,y\rangle = \overline{\langle y,x\rangle}$ Conjugate Symmetric

IP4 : $\langle x,x\rangle \geq 0$ and $\langle x,x\rangle = 0 \Leftrightarrow x = 0$ Positive definite

IP3 $\Rightarrow \langle x,x\rangle \in \mathbb{R}$

A vector space with a specified inner product is called an inner product space

Rmk

① If $\mathbb{F}=\mathbb{R}$, then

IP 3: $\langle x,y\rangle = \langle y,x\rangle$

i.e real inner product is symmetric

② In an inner product space $X$, define a norm on $X$ by

$$\boxed{\|x\| = \sqrt{\langle x,x\rangle}}$$

Warning: We will (but haven't) show that this norm satisfies the axioms for normed space

$$\boxed{\begin{matrix}\text{Inner Product} \\ \text{Space}\end{matrix} \Rightarrow \begin{matrix}\text{Normed} \\ \text{Space}\end{matrix} \Rightarrow \begin{matrix}\text{Metric} \\ \text{Space}\end{matrix}}$$

③ Easy to show $\langle x, y+z\rangle = \langle x,y\rangle + \langle x,z\rangle$

$$\langle x, cy\rangle = \bar{c}\langle x,y\rangle \quad \cdots (*)$$

ie. Conjugate linear in 2nd component

$(*)$ $\because$ $\langle x, cy\rangle = \overline{\langle cy, x\rangle}$

$$= \overline{c\langle y,x\rangle}$$

$$= \bar{c}\,\overline{\langle y,x\rangle}$$

$$= \bar{c}\langle x,y\rangle$$

Also, inductively

$$\left\langle \sum_{i=1}^{k} a_i x_i, y\right\rangle = \sum_{i=1}^{k} a_i \langle x_i, y\rangle$$

$$\left\langle x, \sum_{i=1}^{k} a_i y_i\right\rangle = \sum_{i=1}^{k} \overline{a_i}\langle x, y_i\rangle$$

**Defn** (Orthogonality)

Let $X$ be an inner product space

$x, y \in X$ are said to be orthogonal

(denoted by $x \perp y$) if

$$\langle x, y \rangle = 0$$

Two subsets $A, B \subset X$ are said to

be orthogonal if $\forall x \in A, y \in B$,

$$x \perp y.$$

(Denoted by $A \perp B$)

**Rmk** $x \perp y \Leftrightarrow y \perp x$

**eg of inner product space**

① **Standard inner product in $\mathbb{R}^n$ or $\mathbb{C}^n$**

$$x = (x_1, x_2, \cdots, x_n) \quad y = (y_1, y_2, \cdots, y_n)$$

Define $\boxed{\langle x, y \rangle = \sum_{i=1}^{n} x_i \overline{y_i}}$

Rmk: If $\mathbb{F} = \mathbb{R}$, it is dot product

② $\ell^2 = \{ x = (x_1, x_2, x_3, \cdots) : \sum |x_i|^2 < \infty \}$

Define $\boxed{\langle x, y \rangle = \sum_{i=1}^{\infty} x_i \overline{y_i}}$

The series is convergent by Cauchy-Schwaz inequality

$\Rightarrow$ well-defined

**Rmk** $\|x\| = \sqrt{\langle x, x \rangle} = \sqrt{\sum |x_i|^2}$ Norm from $\ell^2$ comes from inner product

3 $L^2([a,b])$. Define inner product

$$\boxed{\langle f,g\rangle = \int_a^b f(t)\,\overline{g(t)}\,dt}$$

eg. Consider $L^2([0,1])$

$$\|t\| = \sqrt{\langle t,t\rangle} = \sqrt{\int_0^1 t^2\,dt} = \frac{1}{\sqrt{3}}$$

$$\langle t-\tfrac{1}{2}, 1\rangle = \int_0^1 (t-\tfrac{1}{2})\overline{(1)}\,dt$$

$$= 0$$

$$\Rightarrow t-\tfrac{1}{2} \perp 1$$

$L^2$-norm comes from $L^2$-inner product

4 Frobenius Product on $M_{m\times n}(\mathbb{F})$

Let $X = M_{m\times n}(\mathbb{F})$ (the vector space of all $m\times n$ matrixes)

Define $\langle A,B\rangle = tr(\underbrace{B^*A}_{n\times n})$ where

$B^* = \overline{B}^t$,

$B^*$ $n\times m$

$A$ $m\times n$

eg. $\underset{2\times 3}{\begin{bmatrix} 1 & 2+i & i \\ 0 & 3i & 2 \end{bmatrix}^*} = \underset{3\times 2}{\begin{bmatrix} 1 & 0 \\ 2-i & -3i \\ -i & 2 \end{bmatrix}}$

and trace of a square matrix = sum of diagonal

eg $tr\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} = 1+5+9 = 15$

We will verify it is an inner product

IP 1: For any $A, B, C \in X$ $\alpha \in \mathbb{F}$

$$\langle A+B, C\rangle = tr(C^*(A+B))$$
$$= tr(C^*A + C^*B)$$
$$= tr(C^*A) + tr(C^*B)$$
$$= \langle A, C\rangle + \langle B, C\rangle$$

IP 2: $\langle \alpha A, B\rangle = tr(B^*(\alpha A))$
$$= tr(\alpha B^*A)$$
$$= \alpha\, tr(B^*A)$$
$$= \alpha \langle A, B\rangle$$

IP 3: $\overline{\langle A, B\rangle} = \overline{tr(B^*A)}$
$$= tr(\overline{B^*A})$$
$$= tr(\overline{B^*}\,\overline{A})$$
$$= tr(B^t\overline{A})$$

$tr(M)$
$= tr(M^t)$

$$= tr((B^t\overline{A})^t)$$
$$= tr(\overline{A}^t B^{tt})$$
$$= tr(A^*B)$$
$$= \langle B, A\rangle$$

IP4: $A \in M_{m\times n}(\mathbb{F})$

$$\langle A,A\rangle = tr(A^*A) = \sum_{j=1}^{n} (A^*A)_{jj} = \sum_{j=1}^{n}\sum_{i=1}^{m} A^*_{ji}A_{ij} = \sum_{j=1}^{n}\sum_{i=1}^{m} \overline{A_{ij}}A_{ij} = \sum_{j=1}^{n}\sum_{i=1}^{m} |A_{ij}|^2 \geq 0$$

Recall:

$AB = C$

$c_{ij} = \sum a_{ik} b_{kj}$

And $\langle A,A\rangle = 0 \Leftrightarrow |A_{ij}| = 0 \ \forall ij$

$\Leftrightarrow A = 0$

$\therefore$ Frobenius Product is an inner product

Two formulas

Let $X$ be an inner product space

$x, y \in X$. Then

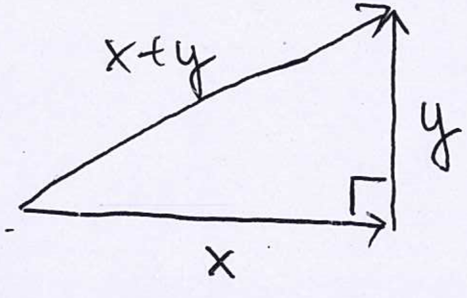
Pythagorean theorem

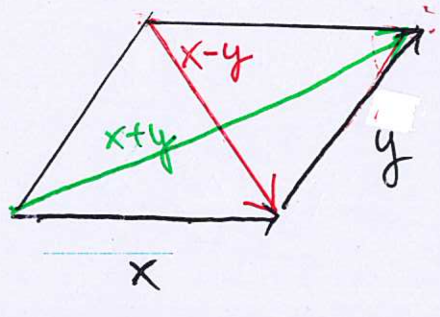
If $x \perp y$, then $\|x+y\|^2 = \|x\|^2 + \|y\|^2$

Parallelogram equality

$\|x+y\|^2 + \|x-y\|^2 = 2(\|x\|^2 + \|y\|^2)$

Picture



**Pf of Pythagorean theorem**

$x \perp y \Rightarrow \langle x, y\rangle = 0$

$$\|x+y\|^2 = \langle x+y, x+y\rangle$$
$$= \langle x,x\rangle + \langle y,x\rangle + \langle x,y\rangle + \langle y,y\rangle$$
$$= \langle x,x\rangle + \langle y,y\rangle$$
$$= \|x\|^2 + \|y\|^2$$

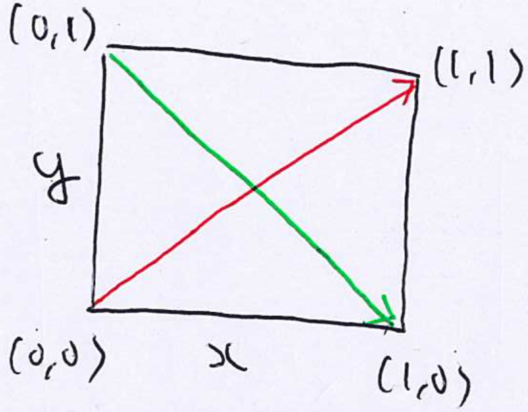
**Eg** Show that $\ell^\infty$-norm on $\mathbb{R}^2$ does not come from an inner product

Recall: $\|(x_1,x_2)\|_\infty = \max\{|x_1|, |x_2|\}$

**Sol** Check that $\ell^\infty$-norm on $\mathbb{R}^2$ fails the parallogram equality

Take $\vec{x} = (1,0)$ $\vec{y} = (0,1)$

$\vec{x}+\vec{y} = (1,1)$ $\vec{x}-\vec{y} = (1,-1)$




$\|x+y\|_\infty^2 + \|x-y\|_\infty^2 = 1^2+1^2 = 2$

$2(\|x\|_\infty^2 + \|y\|_\infty^2) = 2(1^2+1^2) = 4$

$\Rightarrow$ Fails Parallogram equality

$\Rightarrow$ not from inner product

## Two important inequalities

**Cauchy-Schwarz inequality**

Let $X$ be an inner product space $/\mathbb{F}=\mathbb{R}$ or $\mathbb{C}$

$x,y\in X$, then $|\langle x,y\rangle|\leq\|x\|\,\|y\|$

Equality holds $\Leftrightarrow$ $x=\alpha y$ or $y=\alpha x$ for some $\alpha\in\mathbb{F}$

Pf If $y=0$, then

$|\langle x,y\rangle|=0=\|x\|\|y\|$

Equality holds and $y=0x$

If $y\neq0$, consider for any $\alpha\in\mathbb{F}$

$$0\leq\|x-\alpha y\|^2=\langle x-\alpha y,x-\alpha y\rangle$$
$$=\langle x,x\rangle-\langle\alpha y,x\rangle-\langle x,\alpha y\rangle+\langle\alpha y,\alpha y\rangle$$
$$=\|x\|^2-\alpha\langle y,x\rangle-\bar{\alpha}\langle x,y\rangle+|\alpha|^2\|y\|^2$$

Put $\alpha=\dfrac{\langle x,y\rangle}{\langle y,y\rangle}=\dfrac{\langle x,y\rangle}{\|y\|^2}$ ← $y\neq0$ then

$$\|x\|^2-\frac{\langle x,y\rangle}{\|y\|^2}\langle y,x\rangle-\frac{\overline{\langle x,y\rangle}}{\|y\|^2}\langle x,y\rangle+\frac{|\langle x,y\rangle|^2}{\|y\|^4}\|y\|^2\geq0$$

cancel

$$\Rightarrow\ \|x\|^2-\frac{\langle x,y\rangle\overline{\langle x,y\rangle}}{\|y\|^2}\geq0$$

$$\Rightarrow\ \|x\|\,\|y\|\geq|\langle x,y\rangle|$$

If Equality holds, then $\left\|x-\dfrac{\langle x,y\rangle}{\langle y,y\rangle}y\right\|=0$

then $x=\dfrac{\langle x,y\rangle}{\langle y,y\rangle}y\Rightarrow\vec{x}=\alpha y$

If $x=\alpha y$, then $|\langle x,y\rangle|=|\langle\alpha y,y\rangle|=|\alpha|\|y\|^2$

$\|x\|\|y\|=\|\alpha y\|\,\|y\|=|\alpha|\|y\|^2$ ← equal

$\Rightarrow$ Equality holds. Similar proof for $y=\alpha x$

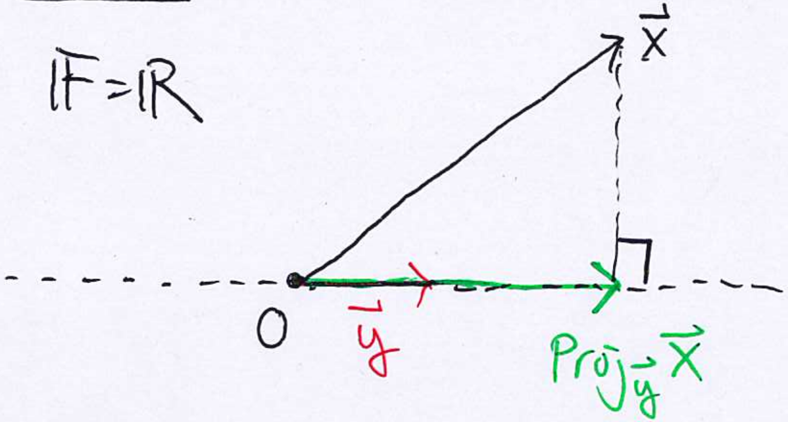
**Q** Why did we set $\alpha = \frac{\langle x,y\rangle}{\langle y,y\rangle}$?

**A** $\frac{\langle x,y\rangle}{\langle y,y\rangle}\vec{y} = \mathrm{Proj}_{\vec{y}}\vec{x}$

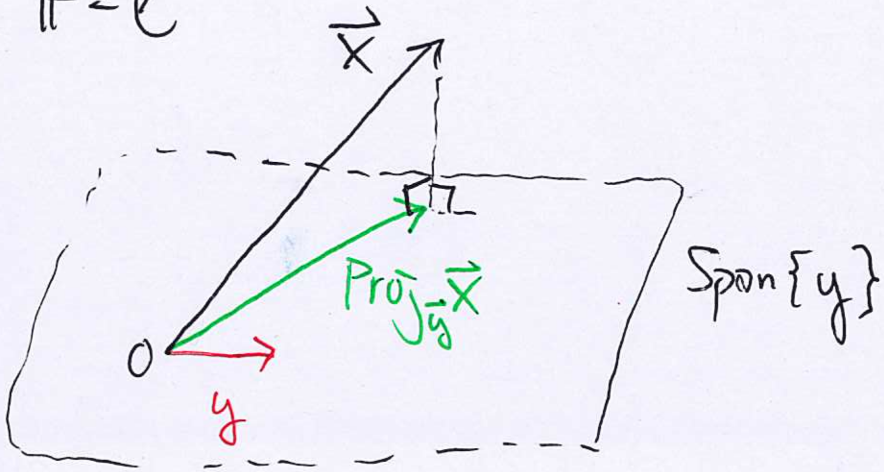
= Orthogonal Projection of $\vec{x}$ onto $\mathrm{Span}\{y\}$

**Picture**

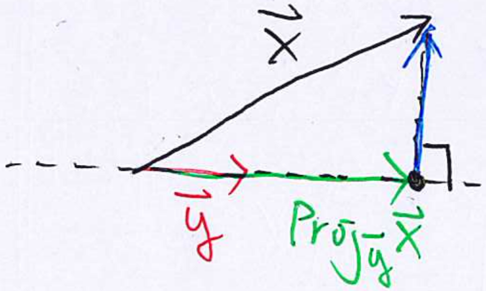
$\mathbb{F} = \mathbb{R}$




$\mathbb{F} = \mathbb{C}$




Formula for $\mathrm{Proj}_{\vec{y}}\vec{x}$




Let $\mathrm{Proj}_{\vec{y}}\vec{x} = \beta\vec{y}$ for some $\beta$

$$\langle \vec{x} - \beta\vec{y}, y\rangle = 0 \iff \langle x,y\rangle - \beta\langle y,y\rangle = 0$$

$$\iff \beta = \frac{\langle x,y\rangle}{\langle y,y\rangle}$$

$$\boxed{\mathrm{Proj}_{\vec{y}}\vec{x} = \frac{\langle x,y\rangle}{\|y\|^2}\, y}$$

$\|x - \alpha y\|^2$ is minimum when $\alpha\vec{y} = \mathrm{Proj}_{\vec{y}}\vec{x}$

$$\iff \alpha = \frac{\langle x,y\rangle}{\langle y,y\rangle}$$

**Triangle inequality** $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$

Let $X$ be an inner product space $x, y \in V$. Then $\|x+y\| \leq \|x\| + \|y\|$

Equality holds $\Leftrightarrow$ $x = \alpha y$ or $y = \alpha x$ for some $\alpha \in \mathbb{R}$, $\alpha \geq 0$

Pf.

$$\|x+y\|^2 = \langle x+y, x+y \rangle$$
$$= \langle x,x \rangle + \langle y,x \rangle + \langle x,y \rangle + \langle y,y \rangle$$
$$= \|x\|^2 + \overline{\langle x,y \rangle} + \langle x,y \rangle + \|y\|^2$$
$$= \|x\|^2 + 2\operatorname{Re}\langle x,y \rangle + \|y\|^2$$
$$\underset{①}{\leq} \|x\|^2 + 2|\langle x,y \rangle| + \|y\|^2$$
$$\underset{②}{\leq} \|x\|^2 + 2\|x\|\|y\| + \|y\|^2 \quad \text{(Cauchy-Schwarz)}$$
$$= (\|x\| + \|y\|)^2$$

Taking square root $\Rightarrow$ $\|x+y\| \leq \|x\| + \|y\|$

Equality holds $\Leftrightarrow$ ① $\operatorname{Re}\langle x,y \rangle = |\langle x,y \rangle|$

② $|\langle x,y \rangle| = \|x\|\|y\|$ (CS)

② $\Leftrightarrow$ $x = \alpha y$ or $y = \alpha x$ for some $\alpha \in \mathbb{F}$

In that case, $\operatorname{Re}\langle x,y \rangle = |\langle x,y \rangle|$

$\Leftrightarrow$ $\langle x,y \rangle \in \mathbb{R}$ and $\langle x,y \rangle \geq 0$

$\Leftrightarrow$ $\alpha \in \mathbb{R}$ and $\alpha \geq 0$

(if $x \neq 0$ or $y \neq 0$)

## Consequence

$$\boxed{\begin{array}{l}\text{Let } X \text{ be an inner product space.} \\ \|x\| = \sqrt{\langle x, x\rangle} \text{ satisfies the} \\ \text{4 axioms for normed space}\end{array}}$$

N1 : $\|x\| \geq 0 \quad \forall x \in X$

N2 : $\|x\| = 0 \Leftrightarrow x = 0$

N3 : $\|\alpha x\| = |\alpha| \, \|x\|$

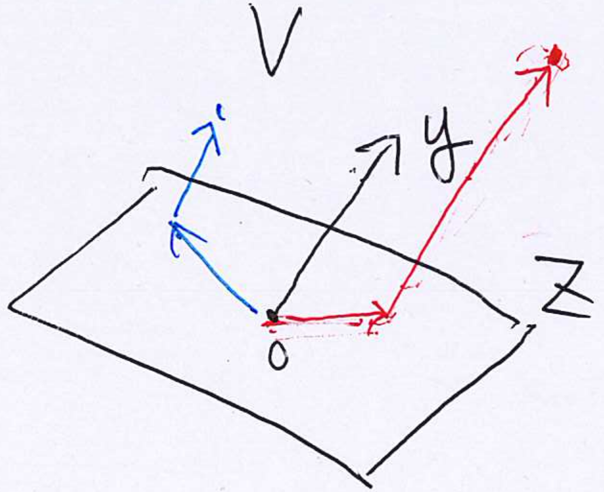
N4 : $\underbrace{\|x+y\| \leq \|x\| + \|y\|}_{\text{We just proved it}}$

Q5 $\dim V = n$ $\quad f, g : V \to \mathbb{R}$

$N(f) = \{x \in V : f(x) = 0\}$ $\boxed{\text{Assignment 3}}$

$N(f) = N(g) = Z$ $\quad \dim Z = n-1$

Show that $\exists c \in \mathbb{R}$ s.t. $f(x) = c\,g(x)$ $\forall x$




Hint ~~Any~~ Let $y \in V \setminus Z$

Then every vector of V can be uniquely expressed as

$z + \alpha y$

$$f(z+\alpha y) = f(z) + \alpha f(y)$$
$$= \alpha \boxed{f(y)} \leftarrow \text{constant}$$
$$g(z+\alpha y) = \alpha \boxed{g(y)} \leftarrow \text{after picking } y$$

Also, can $f(y)$, $g(y)$ be zero?

Q6 $X$ is normed space

$x, y \in X$. Also,

$f(x) = f(y)$ for any bounded linear functional $f$ on $X$

Show that $x = y$

Hint Equivalent to show

If $x \neq y$, then

$\exists$ a bounded linear functional $f$ on $X$ such that

$f(x) \neq f(y)$ (Use Hahn-Banach theorem)